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Ergodic Properties of Infinite-Dimensional Stochastic Systems[1]

M. M. Tropper[2]

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The ergodic properties of two stochastic models Σ_{I} and Σ_{II} are investigated. Each model is described by a field $x(t)$, $t \geqslant 0$, on the lattice $\Gamma = Z^d$, $d < \infty$. For Σ_{I}, $x(t)$ evolves according to the equations

$$dq_s(t) = p_s(t)\,dt$$

$$dp_s(t) = \Big[-\operatorname{grad}\Phi_1(q_s(t)) - \tfrac{1}{2}\beta p_s(t) + \sum_{s' \text{ adjacent to } s} \operatorname{grad}\Phi_2(q_{s'}(t) - q_s(t))\Big]\,dt + dw_s(t) \quad \text{for each } s \in \Gamma$$

where $x_s(t) = (q_s(t), p_s(t)) \in R^2$ is the value taken by $x(t)$ at $s \in \Gamma$. For Σ_{II}, $x(t)$ satisfies

$$dx_s(t) = \Big[-\operatorname{grad}\Phi_1(x_s(t)) + \sum_{s' \text{ adjacent to } s} \operatorname{grad}\Phi_2(x_{s'}(t) - x_s(t))\Big]dt + dw_s(t)$$

where $x_s(t) \in R$ for each $s \in \Gamma$. Here the $\{w_s(t): s \in \Gamma\}$ are independent, one-dimensional Wiener processes, Φ_2 is a bounded interaction between adjacent lattice sites, and the potentials Φ_1 and Φ_2 satisfy appropriate regularity conditions. It is shown that for each model, $x(t)$ is a Markov process on an infinite-dimensional phase space X. The probability measures on X that satisfy the Dobrushin–Lanford–Ruelle (DLR) conditions are stationary for this process and have a mixing property. Moreover, for Σ_{I} any stationary, time-reversal-invariant probability measure that has certain regularity properties must satisfy the DLR conditions.

KEY WORDS: Infinite stochastic system; Markov process; Dobrushin–Lanford–Ruelle conditions; mixing; stationary states; ergodic properties.

1. INTRODUCTION

In a previous paper[1] we made a study of the ergodic behavior of some finite-dimensional stochastic models, and obtained conditions under which these

[1] This paper is based on a portion of the author's Ph.D. thesis.[2]

[2] General Electric Company, Hirst Research Centre, Wembley, Middlesex, England.

models satisfied a mixing property. Here we carry out a similar investigation for two stochastic models whose phase spaces are infinite-dimensional. In each case we seek to describe the stationary states and to find conditions under which the model is mixing with respect to any such state (according to the definition given below).

The models $\Sigma_{\rm I}$ and $\Sigma_{\rm II}$ are described in Section 2. $\Sigma_{\rm I}$ is a Hamiltonian model on an infinite lattice, incorporating bounded interactions between neighboring sites, and with the addition of mutually independent fluctuating forces which act at the individual lattice sites and are representative of thermal reservoirs. The equations of evolution of $\Sigma_{\rm II}$ resemble a lattice version of those of the Ginzburg–Landau theory of superconductivity in their stochastic form, or their analog[3] describing a laser with a continuum of modes.

The defining equations for $\Sigma_{\rm I}$ and $\Sigma_{\rm II}$ are shown in Section 3 to have unique solutions. In Section 4 the corresponding Markov process on the phase space X is described in each case, and the probability measures on X that satisfy the Dobrushin–Lanford–Ruelle (DLR) conditions are shown to be stationary for this process.

Let Σ be a stochastic system with an invariant measure m, and let the variables $x(t) \in X$ describing Σ at time $t \geqslant 0$ have distribution m_t on X. We say that Σ is *mixing with respect to m* if $\lim_{t\to\infty} m_t = m$ whenever m_0 is absolutely continuous with respect to m. In Section 5 (Propositions 5.1 and 5.2) we prove that, subject to some regularity assumptions, both $\Sigma_{\rm I}$ and $\Sigma_{\rm II}$ are mixing with respect to any DLR measure.

Finally, we prove in Section 6 (Proposition 6.1) that any stationary, time-reversal-invariant probability measure for $\Sigma_{\rm I}$ that has certain regularity properties must satisfy the DLR conditions. Thus for this model we have a complete description of the stationary states, together with a mixing property.

The following notation will be used. Z, R, and R^n will denote the positive integers, the real line, and Euclidean n-dimensional space, respectively, and for $n \geqslant 1$ we shall write $|x|$ for the Euclidean norm of $x \in R^n$. If (X, σ, m) is a measure space, and $1 \leqslant p \leqslant \infty$, $L^p(X, m)$ will denote the L^p-class functions on X with respect to m, and 1_A the characteristic function of the set $A \in \sigma$. Finally, for $n \geqslant 1$, $C^2_{\rm com}(R^n)$ and $C^{(2)}(R^n)$ will denote, respectively, the twice continuously differentiable functions with compact support and the continuous, bounded functions having continuous, bounded first and second partial derivatives on R^n.

2. DESCRIPTION OF THE MODELS

We shall now describe the models $\Sigma_{\rm I}$ and $\Sigma_{\rm II}$ with which we shall be concerned. The evolution of each model is given by a field $x(t)$, $t \geqslant 0$, on the lattice $\Gamma = Z^d$, $1 \leqslant d < \infty$.

Model $\Sigma_{\rm I}$: $x(t) = (q(t), p(t))$ is an R^2-valued lattice field, taking the value $x_s(t) = (q_s(t), p_s(t))$ at the site $s \in \Gamma$. The phase space X is $(R^2)^\Gamma$ and the formal field equations are

$$\begin{aligned} dq_s(t) &= p_s(t)\, dt \\ dp_s(t) &= \Big[F(q_s(t)) - \tfrac{1}{2}\beta p_s(t) + \sum_{s'\sim s} g(q_{s'}(t) - q_s(t))\Big]\, dt + dw_s(t) \end{aligned} \tag{1}$$

for each $s \in \Gamma$

where we write $s' \sim s$ to mean that the lattice sites s, s' are adjacent.

Model $\Sigma_{\rm II}$: $x(t)$ is a real-valued field. The phase space X is R^Γ and the formal field equations are given by

$$dx_s(t) = \Big[F(x_s(t)) + \sum_{s'\sim s} g(x_{s'}(t) - x_s(t))\Big]\, dt + dw_s(t) \tag{2}$$

for each $\quad s \in \Gamma$

In Eqs. (1) and (2), F and g are functions from R to R, representing, respectively, a secular force at each lattice site and a bounded interaction between neighboring sites. The $\{w_s(t): s \in \Gamma\}$ are mutually independent, one-dimensional Wiener processes. $\Sigma_{\rm I}$ may represent an infinite-dimensional classical lattice system which undergoes perturbations due to its thermal environment, while Eq. (2) for $\Sigma_{\rm II}$ resembles a lattice version of the equations of the Ginzburg–Landau theory of superconductivity in their stochastic form, or their analog[3] describing a laser with a continuum of modes. Model $\Sigma_{\rm II}$ may thus represent a stochastic system with infinitely many degrees of freedom which is in a state far from thermal equilibrium.

3. EXISTENCE AND UNIQUENESS OF THE EVOLUTIONS

The systems of equations (1) and (2) will be shown to possess unique solutions. The method of proof is similar to that employed in Ref. 4 to obtain the time evolution of an infinite-particle system.

We assume that F and g have the following properties:

(a) F, g are twice continuously differentiable, with derivatives which will be denoted by F', F'', g', g''.

(b) F, F', g, and g' satisfy Lipschitz conditions with corresponding constants α_0, α_1, γ_0, and γ_1, respectively.

(c) There exist $\alpha_2, \gamma_2 < \infty$ such that, for all $y \in R$,

$$|F'(y)| + |F''(y)| \leqslant \alpha_2; \qquad |g(y)| + |g'(y)| + |g''(y)| \leqslant \gamma_2$$

For each $\Lambda \subset \Gamma$, with complement Λ^c, define $X_\Lambda = (R^2)^\Lambda$ for $\Sigma_{\rm I}$, and $X_\Lambda = R^\Lambda$ for $\Sigma_{\rm II}$. Then X is isomorphic with the product $X_\Lambda \times X_{\Lambda^c}$, and we shall write correspondingly $x = (x_\Lambda, x_{\Lambda^c})$ for $x \in X$. The X and X_Λ will be given the product Borel sigma-fields $\mathscr{B}$ and $\mathscr{B}_\Lambda$, respectively.

We consider first the evolution of Σ_{I}, and take $X = (R^2)^\Gamma$. For each $n \geqslant 0$ let Λ_n be the closed cube of side $2n$ in Γ, centered at the origin, and define the field $x^{(n)}(t, x^0)$ for $x^0 = (q^0, p^0)$ in X and $t \geqslant 0$ as follows

$$x^{(n)}_{\Lambda_n^c}(t, x^0) = x^0_{\Lambda_n^c} \qquad \text{for all} \quad t \geqslant 0$$

and for $s \in \Lambda_n$, $x_s^{(n)}(t, x^0) = (q_s^{(n)}(t, x^0), p_s^{(n)}(t, x^0))$ is the unique solution of the equations

$$\begin{aligned} q_s^{(n)}(t) &= q_s^0 + \int_0^t p_s^{(n)}(u)\, du \\ p_s^{(n)}(t) &= p_s^0 + \int_0^t \Big[F(q_s^{(n)}(u)) - \tfrac{1}{2}\beta p_s^{(n)}(u) \\ &\quad + \sum_{s' \sim s, s' \in \Lambda_n} g(q_{s'}^{(n)}(u) - q_s^{(n)}(u)) \Big]\, du + w_s(t) - w_s(0) \end{aligned} \tag{3}$$

[Here and henceforth we omit for brevity the x^0 dependence of $q_s^{(n)}(t), p_s^{(n)}(t)$.] This evolution corresponds to keeping the field at sites outside Λ_n fixed at its initial value $x^0_{\Lambda_n^c}$ and letting that inside Λ_n evolve as a closed system.

Define also a field $\tilde{x}^{(n)}(t, x^0)$ by

$$\tilde{x}^{(n)}_{\Lambda_n^c}(t, x^0) = x^0_{\Lambda_n^c} \qquad \text{for all} \quad t \geqslant 0$$

and for $s \in \Lambda_n$, $\tilde{x}_s^{(n)}(t, x^0) = (\tilde{q}_s^{(n)}(t, x^0), \tilde{p}_s^{(n)}(t, x^0))$ is the unique solution of the equations

$$\begin{aligned} \tilde{q}_s^{(n)}(t) &= q_s^0 + \int_0^t \tilde{p}_s^{(n)}(u)\, du \\ \tilde{p}_s^{(n)}(t) &= p_s^0 + \int_0^t \Big[F(\tilde{q}_s^{(n)}(u)) - \tfrac{1}{2}\beta \tilde{p}_s^{(n)}(u) \\ &\quad + \sum_{s' \sim s} g(\tilde{q}_{s'}^{(n)}(u) - \tilde{q}_s^{(n)}(u)) \Big]\, du + w_s(t) - w_s(0) \end{aligned} \tag{4}$$

Here the field outside Λ_n is kept fixed as before, but that inside Λ_n now evolves under the boundary conditions given by $x^0_{\Lambda_n^c}$.

The field $x(t)$ is obtained as a limit as $n \to \infty$ of the partial evolutions $x^{(n)}(t)$.

Proposition 3.1.[3] For each $s \in \Gamma$, $x_s^{(n)}(t, x^0)$ converges as $n \to \infty$, uniformly over $x^0 \in X$ and t in any finite interval $[0, \tau]$, to a limit which we

[3] An existence theorem for the evolution of another infinite-dimensional stochastic model with interactions has been established by Lang,[5] using a similar method.

call $x_s(t, x^0)$. The set of $\{x_s(t, x^0) = (q_s(t, x^0), p_s(t, x^0)): s \in \Gamma\}$ obey the infinite system of equations (1) for Σ_{I}, and are the unique solution of (1) with initial condition $x_s(0, x^0) = x_s{}^0$ for each $s \in \Gamma$.

Proof. For $\Lambda \subset \Gamma$ and $x = (q, p) \in X$, define $\|q\|_\Lambda = \sup_{s\in\Lambda}|q_s|$, $\|p\|_\Lambda = \sup_{s\in\Lambda}|p_s|$, and $\|x\|_\Lambda = \max(\|q\|_\Lambda, \|p\|_\Lambda)$. It follows from property (b) and Eqs. (3) that, if $r < n$,

$$\|x^{(n)}(t) - x^{(n+1)}(t)\|_{\Lambda_r} \leqslant A \int_0^t \|x^{(n)}(u) - x^{(n+1)}(u)\|_{\Lambda_{r+1}}\, du \tag{5}$$

where $A = \alpha_0 + 4d\gamma_0 + \frac{1}{2}\beta + 1$. Using the boundedness of g, we have also

$$\|x^{(n)}(t) - x^{(n+1)}(t)\|_{\Lambda_n} \leqslant B \int_0^t \|x^{(n)}(u) - x^{(n+1)}(u)\|_{\Lambda_n}\, du + 4d\gamma_2 t$$

where $B = \alpha_0 + \frac{1}{2}\beta + 1$, so that (see Ref. 6, p. 41)

$$\|x^{(n)}(t) - x^{(n+1)}(t)\|_{\Lambda_n} \leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1)$$

It follows by iteration from Eq. (5) that

$$\|x^{(n)}(t) - x^{(n+1)}(t)\|_{\Lambda_r} \leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1)(At)^{n-r}/[(n - r)!] \tag{6}$$

and hence that

$$\sum_{n=r}^{\infty} \|x^{(n)}(t) - x^{(n+1)}(t)\|_{\Lambda_r} \leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1)e^{At} < \infty$$

Thus for each $s \in \Lambda_r$, $\{q_s^{(n)}(t)\}$ and $\{p_s^{(n)}(t)\}$ converge as $n \to \infty$ to limits $q_s(t)$ and $p_s(t)$, respectively, the convergence being uniform over $x^0 \in X$ and $t \in [0, \tau]$ for any $\tau < \infty$. Since $q_s^{(n)}$ and $p_s^{(n)}$ are continuous functions of t with probability 1, the same is true for q_s and p_s. Letting $n \to \infty$ in Eqs. (3) we see that

$$q_s(t) = q_s{}^0 + \int_0^t p_s(u)\, du$$

$$p_s(t) = p_s{}^0 + \int_0^t \Big[F(q_s(u)) - \tfrac{1}{2}\beta p_s(u) + \sum_{s'\sim s} g(q_{s'}(u) - q_s(u))\Big]\, du + w_s(t) - w_s(0) \qquad \text{for each} \quad s \in \Gamma$$

To prove uniqueness, suppose that $\xi(t) \in X$ also satisfies Eqs. (1), with $\xi(0) = x^0$. As above we find that for any r

$$\|\xi(t) - x(t)\|_{\Lambda_r} \leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1)(At)^m/(m!) \qquad \text{for all} \quad m \geqslant 0$$

and hence $\xi(t) = x(t)$. QED

We shall later require the following estimate: for $r < n$, by Eq. (6),

$$\begin{aligned}\|x^{(n)}(t, x^0) - x(t, x^0)\|_{\Lambda_r} &\leqslant \sum_{i=n}^{\infty} \|x^{(i)}(t, x^0) - x^{(i+1)}(t, x^0)\|_{\Lambda_r} \\ &\leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1) \sum_{i=n}^{\infty} \frac{(At)^{i-r}}{(i-r)!} \qquad (7)\end{aligned}$$

Note that, as a consequence of the following lemma, $x(t)$ could also have been obtained as the limit as $n \to \infty$ of $\tilde{x}^{(n)}(t)$.

Lemma 3.1. For each $s \in \Gamma$, $\lim_{n\to\infty}|x_s^{(n)}(t, x^0) - \tilde{x}_s^{(n)}(t, x^0)| = 0$, uniformly for $x^0 \in X$ and $t \in [0, \tau]$.

Proof. This follows from the inequality

$$\|x^{(n)}(t, x^0) - \tilde{x}^{(n)}(t, x^0)\|_{\Lambda_r} \leqslant 4d\gamma_2 B^{-1}(e^{Bt} - 1)\frac{(At)^{n-r}}{(n-r)!} \qquad \text{for any } n \geqslant r$$

which is derived as above. QED

The partial evolutions $x^{(n)}(t, x^0)$, $\tilde{x}^{(n)}(t, x^0)$ for Σ_{II} are defined analogously, and the existence of a unique solution of the field equations (2) is proved in the same way for this model.

4. THE ASSOCIATED MARKOV SEMIGROUPS

Let $B(X, \mathscr{B})$ and $B(X_\Lambda, \mathscr{B}_\Lambda)$ be the Banach spaces of bounded, Borel measurable functions on X and X_Λ, respectively, with the supremum norm $\| \ \|_\infty$. We can consider $B(X_\Lambda, \mathscr{B}_\Lambda)$ as a subspace of $B(X, \mathscr{B})$, consisting of those elements that are cylinder functions based on X_Λ. For each $n \geqslant 0$, positivity-preserving contraction semigroups $\{T_t^{(n)}\}$, $\{\tilde{T}_t^{(n)}\}$ and $\{T_t\}$ can be defined on $B(X, \mathscr{B})$ by

$$T_t^{(n)}f(x^0) = \mathscr{E}f(x^{(n)}(t, x^0))$$

$$\tilde{T}_t^{(n)}f(x^0) = \mathscr{E}f(\tilde{x}^{(n)}(t, x^0))$$

$$T_t f(x^0) = \mathscr{E}f(x(t, x^0))$$

where $\mathscr{E}$ denotes expectation. It follows from Proposition 3.1, Lemma 3.1, and the corresponding results for Σ_{II} that if $g \in B(X_\Lambda, \mathscr{B}_\Lambda)$ for some finite region Λ and g satisfies a Lipschitz condition, then, uniformly on bounded time intervals,

$$\lim_{n\to\infty} \|T_t g - T_t^{(n)}g\|_\infty = 0 = \lim_{n\to\infty} \|T_t g - \tilde{T}_t^{(n)}g\|_\infty \qquad (8)$$

For $x \in X$, $A \in \mathscr{B}$, and $t \geqslant 0$, define $P(t, x, A) = (T_t 1_A)(x)$. Then (see Ref. 7) for any $f \in B(X, \mathscr{B})$,

$$T_t f(x) = \int_X P(t, x, dy) f(y)$$

For each of our models we now describe those probability measures m on X that satisfy the so-called DLR conditions, and show that such a measure corresponds in each case to a stationary distribution for $x(t)$. We make the following assumption:

(d) There exist $\Phi_1, \Phi_2 \colon R \to R$, $\Phi_2(y) = \Phi_2(-y)$, such that

$$F(y) = -\Phi_1'(y) \qquad \text{and} \qquad g(y) = \Phi_2'(y) \qquad \text{for all} \quad y \in R$$

Model Σ_{I}: For each finite region $\Lambda \subset \Gamma$, define

$$\begin{aligned} V_\Lambda(q_\Lambda) &= \sum_{s \in \Lambda} \Phi_1(q_s) + \tfrac{1}{2} \sum_{s, s' \in \Lambda, s' \sim s} \Phi_2(q_{s'} - q_s) \\ \tilde{V}_\Lambda(q_\Lambda | q_{\Lambda^c}) &= V_\Lambda(q_\Lambda) + \sum_{s \in \Lambda, \sigma \in \Lambda^c, \sigma \sim s} \Phi_2(q_\sigma - q_s) \\ &= V_\Lambda(q_\Lambda) + \tilde{W}_\Lambda(q_\Lambda | q_{\Lambda^c}), \quad \text{say} \\ \tilde{H}_\Lambda(x_\Lambda | x_{\Lambda^c}) &= \tilde{V}_\Lambda(q_\Lambda | q_{\Lambda^c}) + \tfrac{1}{2}|p_\Lambda|^2 \end{aligned}$$

For each finite Λ and each $x_{\Lambda^c} \in X_{\Lambda^c}$, suppose that:

(e) $\exp[-\beta \tilde{V}_\Lambda(\ |q_{\Lambda^c})] \in L^1(R^\Lambda)$.

(f) $(\partial \tilde{V}_\Lambda / \partial q_s)(\ |q_{\Lambda^c}) \exp[-\beta \tilde{V}_\Lambda(\ |q_{\Lambda^c})] \in L^1(R^\Lambda)$ for each $s \in \Lambda$.

For $\Pi \subset \Gamma$, let m_Π be the projection of the probability measure m onto $(X_\Pi, \mathscr{B}_\Pi)$. We say that m satisfies the Dobrushin–Lanford–Ruelle (DLR) conditions at inverse temperature β if, for any $f \in L^1(X, m)$ and finite region Λ,

$$\begin{aligned} \int_X f(x)\, dm(x) = \int_{X_{\Lambda^c}} dm_{\Lambda^c}(x_{\Lambda^c}) \int_{X_\Lambda} & f(x_\Lambda, x_{\Lambda^c}) N_\Lambda(x_{\Lambda^c}) \\ & \times \exp[-\beta \tilde{H}_\Lambda(x_\Lambda | x_{\Lambda^c})]\, dx_\Lambda \end{aligned}$$

where

$$[N_\Lambda(x_{\Lambda^c})]^{-1} = \int_{X_\Lambda} \exp[-\beta \tilde{H}_\Lambda(x_\Lambda | x_{\Lambda^c})]\, dx_\Lambda$$

Model Σ_{II}: For $x \in X = R^\Gamma$, define $\tilde{V}_\Lambda(x_\Lambda | x_{\Lambda^c})$ as above, and suppose that for each finite Λ and each $x_{\Lambda^c} \in X_{\Lambda^c}$:

(e′) $\exp[-2\tilde{V}_\Lambda(\ |x_{\Lambda^c})] \in L^1(R^\Lambda)$.

(f′) $[\partial \tilde{V}_\Lambda(\ |x_{\Lambda^c}) / \partial x_s] \exp[-2\tilde{V}_\Lambda(\ |x_{\Lambda^c})] \in L^1(R^\Lambda)$ for each $s \in \Lambda$.

The probability measure m satisfies the DLR conditions if, for any $f \in L^1(X, m)$ and finite region Λ,

$$\int_X f(x)\, dm(x) = \int_{X_{\Lambda^c}} dm_{\Lambda^c}(x_{\Lambda^c}) \int_{X_\Lambda} f(x_\Lambda, x_{\Lambda^c}) N_\Lambda(x_{\Lambda^c}) \times \exp[-2\tilde{V}_\Lambda(x_\Lambda | x_{\Lambda^c})]\, dx_\Lambda$$

where

$$[N_\Lambda(x_{\Lambda^c})]^{-1} = \int_{X_\Lambda} \exp[-2\tilde{V}_\Lambda(x_\Lambda | x_{\Lambda^c})]\, dx_\Lambda$$

Note. Using the compactness of the unit ball in the space of Radon measures on a locally compact Hausdorff space, the existence of DLR measures for Σ_{I} and Σ_{II} can be proved, at least in the case when Φ_2 is bounded. Such measures are not in general unique.

Lemma 4.1. Under assumptions (a)–(f), any DLR measure m defines a stationary distribution for Σ_{I}.

Proof. Let $\varphi \in C^2_{\mathrm{com}}(X_\Lambda)$, $\Lambda \subset \Lambda_n$. Using (c), (e), and (f), we can show, just as in the proof of Proposition 3.1, Ref. 1, that for any fixed $x_{\Lambda_n^c}$ and $t \geqslant 0$,

$$(d/dt) \int_{X_{\Lambda_n}} \tilde{T}_t^{(n)} \varphi(x) \exp[-\beta \tilde{H}_{\Lambda_n}(x_{\Lambda_n} | x_{\Lambda_n^c})]\, dx_{\Lambda_n} = 0$$

Hence $(d/dt) \int_X \tilde{T}_t^{(n)} \varphi(x)\, dm(x) = 0$, so that for any $t \geqslant 0$, $\int_X \tilde{T}_t^{(n)} \varphi(x)\, dm(x) = \int_X \varphi(x)\, dm(x)$, and so by Eq. (8) also $\int_X T_t \varphi(x)\, dm(x) = \int_X \varphi(x)\, dm(x)$. Thus

$$C^2_{\mathrm{com}}(X_\Lambda) \subset \mathscr{L} = \left\{ \varphi \in B(X_\Lambda, \mathscr{B}_\Lambda) : \int_X T_t \varphi(x)\, dm(x) = \int_X \varphi(x)\, dm(x) \quad \text{for all } t \geqslant 0 \right\}$$

Since $\mathscr{L}$ is closed under addition, scalar multiplication, and bounded pointwise limits, it follows from p. 160 of Ref. 7 that $1_E \in \mathscr{L}$ for all $E \in \mathscr{B}_\Lambda$. Hence, by definition of the product topology in X,

$$\int_X P(t, x, A)\, dm(x) = m(A) \qquad \text{for all} \quad A \in \mathscr{B}, \quad t \geqslant 0 \qquad \text{QED}$$

A similar argument shows that, if (a)–(d), (e′), and (f′) are satisfied, and m is a DLR measure for Σ_{II}, then m defines a stationary distribution for Σ_{II}. Thus for both models, any DLR measure m is invariant for the Markov process $x(t)$ with transition probability $P(t, x, A)$, and so the equation

$$T_t f(x) = \int_X P(t, x, dy) f(y)$$

defines $\{T_t\}$ as a semigroup of linear, positivity-preserving contractions on the Hilbert space $\mathscr{H} = L^2(X, m)$ (see Chapter XIII, Par. 1, Theorem 1 of Ref. 8). Let $\| \;\|$ and $\langle \; \rangle$ denote, respectively, the norm and inner product in $\mathscr{H}$.

Lemma 4.2. $\{T_t\}$ is strongly continuous on $\mathscr{H}$.

Proof. Let $f \in C^{(2)}(X_\Lambda)$, where Λ is a finite region. Let $x \in X$, $\tau > 0$. By Eqs. (8), given $\epsilon > 0$ there exists n such that $\Lambda \subset \Lambda_n$ and $|T_t f(x) - T_t^{(n)} f(x)| < \frac{1}{2}\epsilon$ for all $t \in [0, \tau]$. Since $x_{\Lambda_n}^{(n)}(t)$ is a diffusion process and thus stochastically continuous, it follows that $\lim_{t\downarrow 0} T_t^{(n)} f(x) = f(x)$. Hence there exists $\tau_0 < \tau$ such that $|T_t^{(n)} f(x) - f(x)| < \frac{1}{2}\epsilon$ for all $t \leqslant \tau_0$, and so $|T_t f(x) - f(x)| < \epsilon$ for all $t \leqslant \tau_0$. By the dominated convergence theorem,

$$\lim_{t\downarrow 0} \|T_t f - f\|^2 = \lim_{t\downarrow 0} \int_X |T_t f(x) - f(x)|^2 \, dm(x) = 0$$

The domain of strong continuity of $\{T_t\}$ thus includes $\bigcup_{\Lambda \text{ finite}} C^{(2)}(X_\Lambda)$, and so by density is all of $\mathscr{H}$.

5. MIXING

Proposition 5.1. Let F and g satisfy (a)–(f). Then Σ_{I} is mixing with respect to any DLR measure, according to the definition given in Section 1.

Proposition 5.2. Let F and g satisfy (a)–(d), (e′), and (f′). Then Σ_{II} is mixing with respect to any DLR measure.

In this section we shall prove Proposition 5.1. The proof of Proposition 5.2 is similar and will be omitted.

The following lemmas will be needed. The proof of Lemma 5.1 uses results from the theory of finite-dimensional stochastic differential equations and estimates similar to those of Proposition 3.1 (see Chapter 6, Ref. 2). Lemmas 5.2 and 5.3 are proved in the appendix.

Lemma 5.1. Let $f \in C^{(2)}(X_\Lambda)$ and $t \geqslant 0$. Then for $i \in \Lambda_n$, $\Lambda_n \supset \Lambda$, $(\partial/\partial p_i)\tilde{T}_t^{(n)} f$ and $(\partial/\partial p_i) T_t f$ exist and lie in $B(X, \mathscr{B})$, and

$$\lim_{\Lambda\to\infty} \left\| \frac{\partial T_t f}{\partial p_i} - \frac{\partial \tilde{T}_t^{(n)} f}{\partial p_i} \right\|_\infty = 0$$

uniformly for t in any finite interval.

Lemma 5.2. For any finite region Λ, define the differential operators $G_\Lambda{}^1$ and $\tilde{G}_\Lambda{}^2$ by

$$G_\Lambda{}^1 = \frac{1}{2} \exp\left(\frac{1}{2}\beta |p_\Lambda|^2\right) \sum_{i\in\Lambda} \frac{\partial}{\partial p_i} \left[\exp\left(-\frac{1}{2}\beta |p_\Lambda|^2\right) \frac{\partial}{\partial p_i}\right]$$

$$\tilde{G}_\Lambda{}^2 = \sum_{i\in\Lambda} \left[\frac{\partial \tilde{V}_\Lambda(q_\Lambda | q_{\Lambda^c})}{\partial q_i} \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial q_i}\right]$$

Let $\varphi \in C^2_{\text{com}}(X_{\Lambda'})$, $f \in C^{(2)}(X_\Lambda)$, where Λ, Λ' are finite. Then for all $t \geqslant 0$,

$$(d/dt)\langle \varphi, T_t f\rangle = \langle (G^1_{\Lambda'} + \tilde{G}^2_{\Lambda'})\varphi, T_t f\rangle \tag{9}$$

Lemma 5.3. $\mathscr{H}$ is a separable Hilbert space.

Proof of Proposition 5.1. Let m be a DLR measure on $X = (R^2)^\Gamma$, and, as in Section 4, define $\mathscr{H} = L^2(X, m)$. Let $f \in C^{(2)}(X_\Lambda)$, $\Lambda \subset \Lambda_n$. Then $\tilde{T}^{(n)}_t f \in \mathscr{H}$, and

$$\|\tilde{T}^{(n)}_t f\|^2 = \int_{X_{\Lambda_n^c}} \Psi(t, x_{\Lambda_n^c})\, dm_{\Lambda_n^c}(x_{\Lambda_n^c}) \tag{10}$$

where

$$\Psi(t, x_{\Lambda_n^c}) = N_{\Lambda_n}(x_{\Lambda_n^c}) \int_{X_{\Lambda_n}} |\tilde{T}^{(n)}_t f(x)|^2 \exp[-\beta \tilde{H}_{\Lambda_n}(x_{\Lambda_n}|x_{\Lambda_n^c})]\, dx_{\Lambda_n} \tag{11}$$

Using (c), (e), and (f) to differentiate under the integral and integrate by parts in (11), we obtain

$$\frac{\partial}{\partial t}\Psi(t, x_{\Lambda_n^c}) = N_{\Lambda_n}(x_{\Lambda_n^c}) \sum_{i\in\Lambda_n} -\left|\frac{\partial}{\partial p_i}\tilde{T}^{(n)}_t f(x)\right|^2 \exp[-\beta\tilde{H}_{\Lambda_n}(x_{\Lambda_n}|x_{\Lambda_n^c})]\, dx_{\Lambda_n}$$

By Lemma 5.1, given $\tau < \infty$, there exists $M < \infty$ such that $|\partial \tilde{T}^{(n)}_t f(x)/\partial p_i| \leqslant M$ for all $x \in X$, $t \in [0, \tau]$. Thus differentiation under the integral sign in (10) is permissible, with the result that

$$(d/dt)\|\tilde{T}^{(n)}_t f\|^2 = -\sum_{i\in\Lambda_n} \|\partial \tilde{T}^{(n)}_t f/\partial p_i\|^2$$

Hence for any $i \in \Lambda_n$, $t \geqslant 0$,

$$\|\tilde{T}^{(n)}_t f\|^2 - \|f\|^2 \leqslant -\int_0^t \|\partial \tilde{T}^{(n)}_u f/\partial p_i\|^2\, du$$

By Eq. (8) and Lemma 5.1, letting $n \to \infty$,

$$\|T_t f\|^2 - \|f\|^2 \leqslant -\int_0^t \|\partial T_u f/\partial p_i\|^2\, du$$

Since $\|T_t f\|$ is nonincreasing and bounded below, it follows that

$$\int_0^\infty \|\partial T_u f/\partial p_i\|^2\, du < \infty$$

and hence that there exists a sequence $(t_\sigma) \uparrow \infty$ such that

$$\lim_{\sigma\to\infty} \|\partial T_{t_\sigma} f/\partial p_i\| = 0 \tag{12}$$

As in Ref. 9, call such a sequence a $(*)$-sequence. Since $\{T_t f\}_{t\geqslant 0}$ is bounded in

$\mathscr{H}$, it has a weakly convergent subsequence; let $(u_\sigma) \uparrow \infty$ be such that weak-$\lim_{\sigma\to\infty} T_{u_\sigma} f = \gamma \in \mathscr{H}$. Passing to a subsequence if necessary, we may assume (cf. Refs. 2 and 9) that $\lim_{\sigma\to\infty}(u_\sigma - t_\sigma) = 0$ for some (*)-sequence (t_σ). It follows from the strong continuity of $\{T_t\}$ on $\mathscr{H}$ that also

$$\operatorname*{weak-lim}_{\sigma\to\infty} T_{t_\sigma} f = \gamma$$

Let D_i be the differential operator $\partial/\partial p_i$ defined on the domain

$$\bigcup_{\Lambda\,\text{finite},t\geq 0} T_t C^{(2)}(X_\Lambda)$$

which is dense in $\mathscr{H}$. Define W_i on $\bigcup_{\Lambda\,\text{finite}} C^2_{\text{com}}(X_\Lambda)$ by $W_i\varphi(x) = \beta p_i \varphi(x) - \partial\varphi(x)/\partial p_i$. Since D_i and W_i are adjoint, the adjoint operator $D_i{}^*$ exists and extends W_i. If $\varphi \in C^2_{\text{com}}(X_{\Lambda'})$, Λ' finite, then by Eq. (12)

$$\langle\gamma, W_i\varphi\rangle = \lim_{\sigma\to\infty}\langle T_{t_\sigma} f, W_i\varphi\rangle = \lim_{\sigma\to\infty}\langle D_i T_{t_\sigma} f, \varphi\rangle = 0 \qquad (13)$$

Let $\Lambda' = \Pi \bigcup \{i\}$, where Π is finite and does not contain i, and let Γ_i denote $\Gamma\backslash\{i\}$. From Eq. (13) with $\varphi(x_{\Lambda'}) = \varphi_1(x_\Pi)\varphi_2(q_i)\varphi_3(p_i)$, where $\varphi_1 \in C^2_{\text{com}}(X_\Pi)$ and φ_2, $\varphi_3 \in C^2_{\text{com}}(R)$,

$$\int_{X_{\Gamma_i}} dm_{\Gamma_i}(x_{\Gamma_i})\,\varphi_1(x_\Pi) N_{\{i\}}(x_{\Gamma_i}) \int_{R^2} \Big(\gamma(x)\varphi_2(q_i) \times \exp[-\beta\tilde{V}_{\{i\}}(q_i|q_{\Gamma_i})] \times \frac{d}{dp_i}\Big\{\Big[\exp\Big(-\frac{1}{2}\beta p_i{}^2\Big)\Big]\varphi_3(p_i)\Big\}\Big)\,dq_i\,dp_i = 0$$

Let

$$\bar{\gamma}(p_i) = \int_{X_{\Gamma_i}} dm_{\Gamma_i}(x_{\Gamma_i})\,\varphi_1(x_\Pi) N_{\{i\}}(x_{\Gamma_i}) \times \int_R \{\gamma(x)\varphi_2(q_i)\exp[-\beta\tilde{V}_{\{i\}}(q_i|q_{\Gamma_i})]\}\,dq_i$$

$\bar{\gamma}$ is defined for any φ_1, φ_2 of the above form outside a Lebesgue-null set of p_i, and $\int_{-\infty}^{\infty}\bar{\gamma}(p_i)(d\psi/dp_i)\,dp_i = 0$ for any $\psi \in C^2_{\text{com}}(R)$. Hence (see Ref. 1, Lemma 3.2), $\bar{\gamma}$ is a constant. Using the arbitrariness of Π, φ_1, and φ_2, the fact that m is absolutely continuous with respect to m_{Γ_i} × (Lebesgue measure on R^2), and the fact that $\gamma(x_{\Gamma_i}, q_i, \ )$ is locally integrable on R for all (x_{Γ_i}, q_i) outside an m_{Γ_i} × (Lebesgue)-null set, we can prove by an argument similar to that of Lemma 3.1, Ref. 1 that γ is independent of p_i m-almost everywhere. By the arbitrariness of i, we conclude that $\gamma = \gamma(q)$ is independent of p.

We have thus proved that weak-$\lim_{\sigma\to\infty} T_{u_\sigma} f = \gamma$, a p-independent function. Let $t > 0$. Arguing as in the proof of Proposition 3.1, Ref. 1, we find that $T_t\gamma = \text{weak-lim}_{\sigma\to\infty} T_{t+u_\sigma}f$ is also p-independent. Let $\{T_t\}$ have infinitesimal generator Q. The dual semigroup $\{T_t^*\}$ is weakly continuous, and hence, by Lemma 5.3, also strongly continuous, with infinitesimal generator Z satisfying $ZT_t^*g = Q^*T_t^*g$ for any $g \in \text{Dom}\,(Z)$, $t \geqslant 0$. For any finite region Λ' and $\varphi \in C^2_{\text{com}}(X_{\Lambda'})$, $t > 0$,

$$\begin{aligned}(d/dt)\,\langle\varphi, T_t\gamma\rangle &= \langle Q^*T_t^*\varphi, \gamma\rangle \\ &= \lim_{\sigma\to\infty}\langle\varphi, QT_{t+u_\sigma}f\rangle = \lim_{\sigma\to\infty}\{(d/dt)\,\langle\varphi, T_{t+u_\sigma}f\rangle\} \\ &= \lim_{\sigma\to\infty}\langle(G^1_{\Lambda'} + \tilde{G}^2_{\Lambda'})\varphi, T_{t+u_\sigma}f\rangle \qquad \text{by Lemma 5.2} \\ &= \langle(G^1_{\Lambda'} + \tilde{G}^2_{\Lambda'})\varphi, T_t\gamma\rangle = \langle\tilde{G}^2_{\Lambda'}\varphi, T_t\gamma\rangle,\end{aligned}$$

since $T_t\gamma$ is p-independent. Let $\Lambda' = \Lambda_n \bigcup \Pi$, where Π is disjoint from Λ_n, and let φ have the form $\varphi(x_{\Lambda'}) = \varphi_1(q_\Pi)\varphi_2(p_\Pi)\varphi_3(q_{\Lambda_n})\varphi_4(p_{\Lambda_n})$, where φ_1, $\varphi_2 \in C^2_{\text{com}}(R^\Pi)$, φ_3, $\varphi_4 \in C^2_{\text{com}}(R^{\Lambda_n})$, and

$$\int_{R^{\Lambda_n}} [\exp(-\tfrac{1}{2}\beta|p_{\Lambda_n}|^2)]\varphi_4(p_{\Lambda_n})\,dp_{\Lambda_n} = 0 \tag{14}$$

Then $\langle\varphi, T_t\gamma\rangle = 0$ for all $t \geqslant 0$, and it follows that

$$\begin{aligned}0 &= \langle\tilde{G}^2_{\Lambda'}\varphi, \gamma\rangle \\ &= \sum_{i\in\Lambda_n}\int_X \varphi_1(q_\Pi)\varphi_2(p_\Pi)\varphi_4(p_{\Lambda_n})p_i\gamma(q) \\ &\quad\times\left[\beta\frac{\partial\tilde{V}_{\Lambda_n}}{\partial q_i}\varphi_3(q_{\Lambda_n}) - \frac{\partial\varphi_3}{\partial q_i}\right]dm(x)\end{aligned}$$

using (14) and integrating by parts in the variables p_i, $i \in \Lambda_n$. Hence by the arbitrariness in the choice of φ_4, for each $i \in \Lambda_n$,

$$0 = \int_X \varphi_1(q_\Pi)\varphi_2(p_\Pi)\gamma(q)\left(\frac{\partial\varphi_3}{\partial q_i} - \beta\frac{\partial\tilde{V}_{\Lambda_n}}{\partial q_i}\varphi_3\right)dm(x) \tag{15}$$

Let $r < n$. From Eq. (15), for each $k \in \Lambda_r$,

$$0 = \int_X \varphi_1(q_\Pi)\varphi_2(p_\Pi)\gamma(q)\left[\frac{\partial\varphi_3}{\partial q_k} - \beta\frac{\partial V_{\Lambda_n}(q_{\Lambda_n})}{\partial q_k}\varphi_3\right]dm(x)$$

and hence

$$0 = \int_{R^{\Lambda_n}} \tilde{\gamma}(q_{\Lambda_n})\frac{\partial}{\partial q_k}\{\varphi_3(q_{\Lambda_n})\exp[-\beta V_{\Lambda_n}(q_{\Lambda_n})]\}\,dq_{\Lambda_n}$$

where

$$\begin{aligned}\tilde{\gamma}(q_{\Lambda_n}) = \int_{X_{\Lambda_n^c}} & N_{\Lambda_n}(x_{\Lambda_n^c})\varphi_1(q_\Pi) \\ &\times\varphi_2(p_\Pi)\gamma(q)\exp[-\beta\tilde{W}_{\Lambda_n}(q_{\Lambda_n}|q_{\Lambda_n^c})]\,dm_{\Lambda_n^c}(x_{\Lambda_n^c})\end{aligned} \tag{16}$$

is defined for any φ_1, φ_2 of the above form outside a Lebesgue-null set of q_{Λ_n}, and locally integrable on R^{Λ_n}. Thus by (a), for any $\psi \in C^2_{\text{com}}(R^{\Lambda_n})$ and each $k \in \Lambda_r$,

$$0 = \int_{R^{\Lambda_n}} \tilde{\gamma}(q_{\Lambda_n}) \frac{\partial \psi(q_{\Lambda_n})}{\partial q_k} \, dq_{\Lambda_n}$$

so that, from Lemma 3.2, Ref. 1, $\tilde{\gamma}$ is q_{Λ_r}-independent Lebesgue-almost everywhere. Let $\Lambda_n{}^r = \Lambda_n \backslash \Lambda_r$. Since $\gamma \in \mathscr{H} \subset L^1(X, m)$, $\gamma(q_{\Lambda_n^c}, q_{\Lambda_n^r})$ is locally integrable on R^{Λ_r} for all $(q_{\Lambda_n^c}, q_{\Lambda_n^r})$ outside an $m_{\Lambda_n^c} \times$ (Lebesgue)-null set N_1. Also, because of the arbitrariness of Π, φ_1, φ_2 in (16), there is an $m_{\Lambda_n^c} \times$ (Lebesgue)-null set N_2 such that for all $(q_{\Lambda_n^c}, q_{\Lambda_n^r}) \in N_2{}^c$,

$$\int_I \gamma(q) \, dq_{\Lambda_r} = \int_J \gamma(q) \, dq_{\Lambda_r}$$

for any half-open intervals I and J in R^{Λ_r} having rational end points and finite equal Lebesgue measures. Since m is absolutely continuous with respect to $m_{\Lambda_n^c} \times$ [Lebesgue measure on $(R^2)^{\Lambda_n}$], we can deduce as in the proof of Lemma 3.1, Ref. 1 that $\gamma(q)$ is independent of q_{Λ_r} m-almost everywhere.

As r was arbitrary, γ must be a constant, and it follows from the invariance of m that

$$\gamma = \langle f, 1 \rangle = \int_X f(x) \, dm(x)$$

Finally, suppose that weak-$\lim_{t \to \infty} T_t f \neq \langle f, 1 \rangle$. By weak sequential compactness there must exist a sequence $(u_\sigma{}') \uparrow \infty$ such that weak-$\lim_{\sigma \to \infty} T_{u_\sigma'} f = \eta \neq \langle f, 1 \rangle$. We have, however, shown this to be impossible. Hence

$$\lim_{t \to \infty} \int_X T_t f(x) h(x) \, dm(x) = \int_X f(x) \, dm(x) \int_X h(x) \, dm(x) \tag{17}$$

for any $h \in \mathscr{H}$ and $f \in C^{(2)}(X_\Lambda)$, Λ finite.

Density arguments now show that Eq. (17) holds for any $f \in B(X, \mathscr{B})$, $h \in L^1(X, m)$, and the conclusion of Proposition 5.1 follows.

6. THE INVARIANT MEASURES FOR Σ_{I}

We have seen in Section 4 that the measures on X that satisfy the DLR conditions define stationary distributions for both Σ_{I} and Σ_{II}. In this section we shall prove that, for model Σ_{I}, any stationary distribution that is time-reversal invariant and satisfies certain additional requirements must be given by a DLR measure.

Define a time-reversal operator ρ on $B(X, \mathscr{B})$ by

$$\rho f(q, p) = f(q, -p)$$

Proposition 6.1. Let the probability measure m on X be stationary for Σ_{I} and satisfy:

(i) $\int_X \rho f(x)\, dm(x) = \int_X f(x)\, dm(x)$ for all $f \in B(X, \mathscr{B})$.

(ii) $\int_X (|p_s| + |q_s|)\, dm(x) < \infty$ for each $s \in \Gamma$.

(iii) For each finite region Λ, and $x_{\Lambda^c} \in X_{\Lambda^c}$, the conditional probability measure $m(\ |x_{\Lambda^c})$ exists on X_Λ and is absolutely continuous with respect to Lebesgue measure on X_Λ, with corresponding Radon–Nikodym derivative $\tilde{m}_\Lambda(x_\Lambda|x_{\Lambda^c})$.

(iv) $\tilde{m}_\Lambda(x_\Lambda|x_{\Lambda^c}) > 0$ for all $x \in X$, and for any finite Λ and Π with $\Lambda \subset \Pi$,[4]

$$\tilde{m}_\Pi(x_\Pi|x_{\Pi^c}) = \tilde{m}_\Lambda(x_\Lambda|x_{\Lambda^c}) \int_{X_\Lambda} \tilde{m}_\Pi(\xi_\Lambda, x_{\Pi\setminus\Lambda}|x_{\Pi^c})\, d\xi_\Lambda \tag{18}$$

Then m satisfies the DLR conditions.

Note on Condition (18). Let $\mathscr{E}_m(\ |\mathscr{B}_{\Lambda^c})$ denote the conditional expectation given the sigma-algebra $\mathscr{B}_{\Lambda^c} \subset \mathscr{B}$, for the measure m. Then for each $A \in \mathscr{B}$, $x \in X$,

$$\begin{aligned}\mathscr{E}_m(1_A|\mathscr{B}_{\Lambda^c})(x) &= \int_X 1_A(\xi_\Lambda, x_{\Lambda^c})\tilde{m}_\Lambda(\xi_\Lambda|x_{\Lambda^c})\, d\xi_\Lambda \\ &= Q_\Lambda(x, A) \quad \text{say}\end{aligned}$$

Q_Λ is thus a version of the conditional probability of m with respect to $\mathscr{B}_{\Lambda^c}$ for each finite Λ (cf. Ref. 10), and the following consistency condition must therefore hold: For any finite Λ, Π with $\Lambda \subset \Pi$,

$$(Q_\Pi Q_\Lambda)(x, A) = Q_\Pi(x, A) \qquad \text{for all } A \in \mathscr{B} \text{ and } m\text{-almost all } x \in X \tag{19}$$

where

$$(Q_\Pi Q_\Lambda)(x, A) = \int_X Q_\Lambda(y, A) Q_\Pi(x, dy)$$

If Eq. (19) is required to hold everywhere, rather than just m-almost everywhere, (18) follows.

Proof of Proposition 6.1. Let $f \in C^{(2)}(X_{\Lambda'})$, $\Lambda' \subset \Lambda_{n-1}$. For any $t > 0$,

$$\begin{aligned}0 &= \int_X [T_t f(x) - f(x)]\, dm(x) \\ &= \int_X [T_t f(x) - T_t^{(n)} f(x)]\, dm(x) + \int_X [T_t^{(n)} f(x) - f(x)]\, dm(x)\end{aligned} \tag{20}$$

[4] It is of interest to note the equivalence between DLR and KMS conditions, which has been established for an infinite conservative system in Ref. 10.

By Eq. (7) and the Lipschitz property of f, $\|T_t f - T_t^{(n)} f\|_\infty$ is $o(t)$ as $t \downarrow 0$. Hence from Eq. (20), using (ii) to differentiate under the integral sign,

$$0 = \lim_{t \downarrow 0} t^{-1} \int_X [T_t^{(n)} f(x) - f(x)]\, dm(x) = (d/dt) \int_X T_t^{(n)} f(x)\, dm(x)|_{t=0}$$

$$= \int_X (G^1_{\Lambda'} - \tilde{G}^2_{\Lambda'}) f(x)\, dm(x) \tag{21}$$

For any $h \in C^{(2)}(X_{\Lambda'})$, $\rho(G^1_{\Lambda'} h) = G^1_{\Lambda'}(\rho h)$ and $\rho(\tilde{G}^2_{\Lambda'} h) = -\tilde{G}^2_{\Lambda'}(\rho h)$. Thus by (i) and Eq. (21), for each $f \in C^{(2)}(X_{\Lambda'})$,

$$\int_X G^1_{\Lambda'} f(x)\, dm(x) = 0 \tag{22}$$

$$\int_X \tilde{G}^2_{\Lambda'} f(x)\, dm(x) = 0 \tag{23}$$

Let Λ be a finite region, and define

$$\mu_\Lambda(x_\Lambda) = \int_{X_{\Lambda^c}} \tilde{m}_\Lambda(x_\Lambda | x_{\Lambda^c})\, dm_{\Lambda^c}(x_{\Lambda^c})$$

Then $dm_\Lambda(x_\Lambda) = \mu_\Lambda(x_\Lambda)\, dx_\Lambda$, and

$$\bar{\mu}_\Lambda(p_\Lambda) = \int_{R^\Lambda} f_1(q_\Lambda) \mu_\Lambda(q_\Lambda, p_\Lambda)\, dq_\Lambda$$

is defined for any $f_1 \in C^{(2)}(R^\Lambda)$ outside a Lebesgue-null set of p_Λ. From Eq. (22), with $\Lambda' = \Lambda$,

$$\int_{R^\Lambda} \bar{\mu}_\Lambda(p_\Lambda) G_\Lambda{}^1 f_2(p_\Lambda)\, dp_\Lambda = 0 \qquad \text{for any} \quad f_2 \in C^{(2)}(R^\Lambda)$$

and hence $\bar{\mu}_\Lambda$ is a stationary density for the Gaussian diffusion process on R^Λ whose Langevin equation is

$$dp_\Lambda(t) = -\tfrac{1}{2}\beta p_\Lambda(t)\, dt + dw_\Lambda(t)$$

[where $w_\Lambda(t)$ is an R^Λ-valued Wiener process]. Since[11] this process has the unique stationary density const $\times \exp(-\frac{1}{2}\beta |p_\Lambda|^2)$, it follows that

$$\bar{\mu}_\Lambda(p_\Lambda) \exp(\tfrac{1}{2}\beta |p_\Lambda|^2)$$

is constant almost everywhere, and hence from Lemma 3.1 of Ref. 1, that $\mu_\Lambda(q_\Lambda, p_\Lambda) = \nu_\Lambda(q_\Lambda) \exp(-\frac{1}{2}\beta |p_\Lambda|^2)$ almost everywhere (Lebesgue), for some measurable function ν_Λ on R^Λ.

Thus, since Λ was an arbitrary finite region, the measure m on $(R^2)^\Gamma = R^\Gamma \times R^\Gamma$ is the product of a configurational part $m^{(q)}$ and a momentum part

$m^{(p)}$, and $m^{(p)}$ is the product on $\prod_{s\in\Gamma} R$ of the Gaussian probability measures with densities $(2\pi/\beta)^{-1/2}\exp(-\frac{1}{2}\beta p^2)$ on each copy of R. It follows that

$$\tilde{m}_\Lambda(x_\Lambda|x_{\Lambda^c}) = \tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c})\exp(-\tfrac{1}{2}\beta|p_\Lambda|^2) \tag{24}$$

for some measurable p-independent function $\tilde{\nu}_\Lambda$.

Let $\theta\in C^2_{\rm com}(X_\Pi)$, φ_1, $\varphi_2\in C^2_{\rm com}(R^\Lambda)$, where Λ, Π are finite disjoint regions. From Eq. (23) with $\Lambda' = \Lambda\bigcup\Pi$ and f of the form $f(x_{\Lambda'}) = \theta(x_\Pi)\varphi_1(q_\Lambda)\varphi_2(p_\Lambda)$ we find, using the arbitrariness of φ_2, that for each $i\in\Lambda$,

$$\int_{X_{\Lambda^c}} dm_{\Lambda^c}(x_{\Lambda^c})\,\theta(x_\Pi) \times\int_{R^\Lambda}\tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c})\left[\beta\frac{\partial\tilde{V}_\Lambda(q_\Lambda|q_{\Lambda^c})}{\partial q_i}\varphi_1(q_\Lambda) - \frac{\partial\varphi_1}{\partial q_i}\right]dq_\Lambda = 0 \tag{25}$$

Let $\Lambda^0 = \{i\in\Lambda\text{: if } s\in\Gamma,\ s\sim i, \text{ then } s\in\Lambda\}$; $\partial\Lambda = \Lambda\backslash\Lambda^0$. From Eq. (25), if $i\in\Lambda^0$,

$$\int_{X_{\Lambda^c}} dm_{\Lambda^c}(x_{\Lambda^c})\,\theta(x_\Pi)\int_{R^\Lambda}\left(\{\exp[\beta V_\Lambda(q_\Lambda)]\}\tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c}) \times\frac{\partial}{\partial q_i}\{\varphi_1(q_\Lambda)\exp[-\beta V_\Lambda(q_\Lambda)]\}\right)dq_\Lambda = 0 \tag{26}$$

Thus, using (a), $\int_{R^\Lambda}\bar{\nu}_\Lambda(q_\Lambda)[\partial\psi(q_\Lambda)/\partial q_i]\,dq_\Lambda = 0$ for any $i\in\Lambda^0$, $\psi\in C^2_{\rm com}(R^\Lambda)$, where

$$\bar{\nu}_\Lambda(q_\Lambda) = \{\exp[\beta V(q_\Lambda)]\}\int_{X_{\Lambda^c}}\theta(x_\Pi)\tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c})\,dm_{\Lambda^c}(x_{\Lambda^c})$$

is locally integrable on R_Λ.

Hence by Lemma 3.2, Ref. 1, $\bar{\nu}_\Lambda(q_\Lambda)$ is q_{Λ^0}-independent almost everywhere. Using the arbitrariness of Π and θ, we can now show, by an argument similar to that used in the proof of Proposition 5.1 to prove the q_{Λ_r} independence of $\gamma(q)$, that $\{\exp[\beta V_\Lambda(q_\Lambda)]\}\tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c})$ is q_{Λ^0}-independent m-almost everywhere. It follows that

$$\tilde{\nu}_\Lambda(q_\Lambda|q_{\Lambda^c}) = \{\exp[-\beta V_\Lambda(q_\Lambda)]\}\Psi_\Lambda(q_{\partial\Lambda}|q_{\Lambda^c}) \tag{27}$$

for some Ψ_Λ that is independent of q_{Λ^0}.

Finally, substituting Eqs. (24) and (27) into Eq. (18) for regions Λ, Π, with $\Lambda\subset\Pi^0$, and using the identity

$$V_\Pi(q_\Pi) = V_\Lambda(q_\Lambda) + V_{\Pi\backslash\Lambda}(q_{\Pi\backslash\Lambda}) + \tilde{W}_\Lambda(q_\Lambda|q_{\Lambda^c})$$

we find that

$$\tilde{m}_\Lambda(x_\Lambda|x_{\Lambda^c}) = \frac{\exp[-\beta\tilde{H}_\Lambda(x_\Lambda|x_{\Lambda^c})]}{\int_{X_\Lambda}\exp[-\beta\tilde{H}_\Lambda(x_\Lambda|x_{\Lambda^c})]\,dx_\Lambda}$$

and hence that m satisfies the DLR conditions. QED

7. CONCLUDING REMARKS

We have shown not only that measures satisfying the DLR conditions are stationary for both our models, but also that under certain circumstances such measures are the only stationary states for Σ_{I}. The DLR conditions describe states that possess internal local stability.[12] They are generally assumed to describe equilibrium states of infinite-dimensional classical systems, both stochastic and mechanical. We see that they may also be steady states of systems, such as lasers, which undergo pumping and are thus not in thermal equilibrium.

In an extension of our previous work on finite-dimensional systems, we have also proved that models Σ_{I} and Σ_{II} are mixing with respect to any DLR measure.

Results similar to ours on stationarity and mixing properties of DLR states are obtained in Ref. 13 for a classical lattice spin system whose evolution is given by a Markov process with detailed balance, and in Refs. 4, 14, and 15 for an infinite system of classical particles with a conservative evolution. Although time-reversal invariance is not imposed there, the conditions under which in Ref. 14 a stationary state is shown to be DLR are otherwise more restrictive than those we require in Proposition 6.1.

APPENDIX

Proof of Lemma 5.2. Let $n > 0$ be such that $\Lambda \cup \Lambda' \subset \Lambda_n$. Now,

$$\langle \varphi, \tilde{T}_t^{(n)} f \rangle = \int_{X_{\Lambda_n^c}} dm_{\Lambda_n^c}(x_{\Lambda_n^c})\, \Phi(t, x_{\Lambda_n^c}) \tag{A1}$$

where

$$\Phi(t, x_{\Lambda_n^c}) = N_{\Lambda_n}(x_{\Lambda_n^c}) \int_{X_{\Lambda_n}} \tilde{T}_t^{(n)} f(x) \varphi(x) \exp[-\beta \tilde{H}_{\Lambda_n}(x_{\Lambda_n} | x_{\Lambda_n^c})]\, dx_{\Lambda_n}$$

Using (c), (e), and (f) to differentiate under the integral and integrate by parts,

$$\frac{\partial}{\partial t} \Phi(t, x_{\Lambda_n^c}) = N_{\Lambda_n}(x_{\Lambda_n^c}) \int_{X_{\Lambda_n}} \tilde{T}_t^{(n)} f(x) [(G_{\Lambda'}^1 + \tilde{G}_{\Lambda'}^2)\varphi](x)$$
$$\times \exp[-\beta \tilde{H}_{\Lambda_n}(x_{\Lambda_n} | x_{\Lambda_n^c})]\, dx_{\Lambda_n}$$

Now $(G_{\Lambda'}^1 + G_{\Lambda'}^2)\varphi$ is bounded on X, and $|\tilde{T}_t^{(n)} f(x)| \leqslant \|f\|_\infty$ for all $x \in X$, $t \geqslant 0$. Hence we may differentiate under the integral in (A1), obtaining

$$(d/dt)\langle \varphi, \tilde{T}_t^{(n)} f \rangle = \int_X dm(x)\, \tilde{T}_t^{(n)} f(x) [(G_{\Lambda'}^1 + \tilde{G}_{\Lambda'}^2)\varphi](x)$$
$$= \langle (G_{\Lambda'}^1 + \tilde{G}_{\Lambda'}^2)\varphi, \tilde{T}_t^{(n)} f \rangle$$

Equation (9) follows on letting $n \to \infty$. QED

Proof of Lemma 5.3. For each $n \geqslant 0$, $L^2(X_{\Lambda_n}, m_{\Lambda_n}) = \mathscr{H}_n$ is a separable subspace of $\mathscr{H}$, consisting of cylinder functions based on Λ_n. Let S_n be a countable dense set for $\mathscr{H}_n$, and define $S = \bigcup_{n=0}^{\infty} S_n$. We shall show that the countable set of all finite sums of rational multiples of elements in S is dense in $\mathscr{H}$.

Let $f \in \mathscr{H}$, $\epsilon > 0$. There exists a simple function $\bar{f} = \sum_{i=1}^{N} r_i 1_{B_i}$, where the r_i are rationals and $B_i \in \mathscr{B}$, $i = 1,..., N$, such that $\|f - \bar{f}\| < \frac{1}{3}\epsilon$. Now by definition $\mathscr{B}$ is the sigma-field generated by the semiring $\mathscr{C}$ of cylinder sets of X. Hence for each $i \leqslant N$ there is a set R_i, consisting of a finite union of disjoint sets of $\mathscr{C}$, such that $m(B_i \,\Delta\, R_i) \leqslant (\epsilon/3Nr_i)^2$, where Δ denotes the symmetric difference. It follows that $\|1_{B_i} - 1_{R_i}\| \leqslant \epsilon/(3N|r_i|)$. Also, since $1_{R_i} \in \mathscr{H}_{n(i)}$ for some $n(i) < \infty$, there exists $\varphi_i \in S_{n(i)} \subset S$ such that $\|1_{R_i} - \varphi_i\| \leqslant \epsilon/(3N|r_i|)$. Thus

$$\begin{aligned}\left\| f - \sum_{i=1}^{N} r_i\varphi_i \right\| &\leqslant \|f - \bar{f}\| + \sum_{i=1}^{N} |r_i| \, \|1_{B_i} - 1_{R_i}\| \\ &\quad + \sum_{i=1}^{N} |r_i| \, \|1_{R_i} - \varphi_i\| \\ &\leqslant 3(\epsilon/3) = \epsilon \quad \text{QED}\end{aligned}$$

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